# An Axiomatic and Data Driven View on the EPK Paradox

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# Motivation

- Pricing kernel (PK)
  - Consumption based models
    - marginal rate of consumption substitution
  - Arbitrage free models
    - Radon-Nikodym derivative of the physical measure w.r.t. the risk neutral measure Risk Neutral Valuation PK - Black-Scholes
- Empirical pricing kernel (EPK)
  - Any estimate of the PK
  - EPK paradox - locally increasing EPK

# EPK Paradox

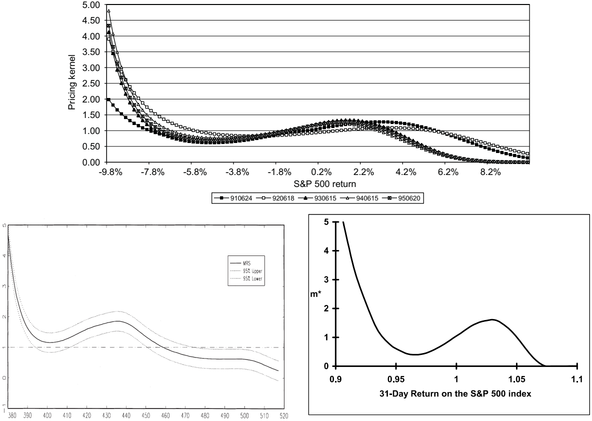


Figure 1: EPK's: Engle and Rosenberg (2002), Ait-Sahalia and Lo (2000), Brown and Jackwerth (2004)

# EPK Paradox

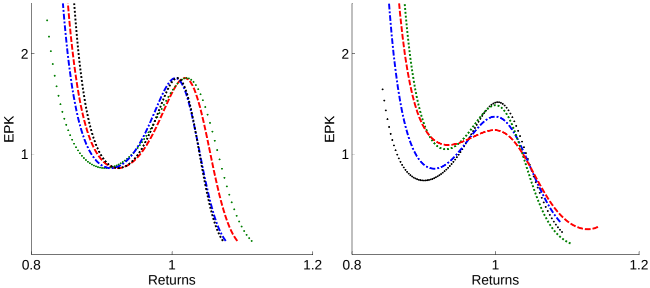


Figure 2: EPK's for various maturities (left) and different estimation dates for fixed maturity 1M (right), Grith et al. (2010)

# EPK Paradox

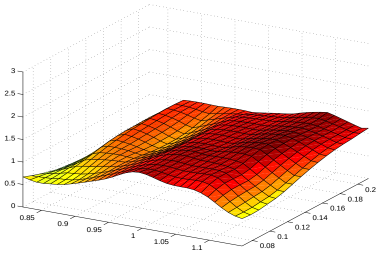


Figure 3: EPK's across moneyness $\kappa$ and maturity $\tau$ for DAX from 20010101 – 20011231, Giacomini and Härdle (2008)

# EPK Paradox

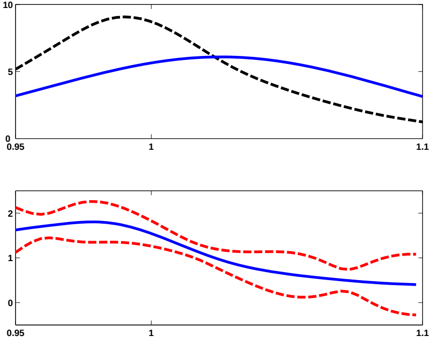


Figure 4: Upper panel: estimated risk neutral density $\widehat{q}$ and historical density $\widehat{p}$. Lower panel: EPK and 95% uniform confidence bands on 20080228, Härdle et al. (2010)

# Objectives

- Pricing kernel derivation
  - Adjust individual and aggregate preferences
  - State-dependent (state variable: market return)
  - Simulation study
- Fitting EPK's
  - Identifiability of parameters
  - Empirical study

# Research Questions

- How to modify standard expected utility theory to rationalize the EPK paradox?
- How well can 'observed' EPK's be fitted?
- How sensitive are results with respect to the preference parameters?
- How to estimate the time variation of estimated parameters/functions?

# Outline

1. Motivation ✓
2. Microeconomic Framework
3. Pricing Kernel
4. Fitting EPK's
5. Empirical Study
6. Statistical Properties
7. Conclusions

# Assumptions

- Financial markets
  - Finite investment time horizon $[0, T]$ and $r$ risk free interest rate
  - Risky asset with prices $\{S_t\}_{0 \le t \le T}$ and return $R_T = S_T / S_0$
  - Arbitrage free market, at least one equivalent martingale measure with density $\pi$
- $m$ Consumers
  - Endowment $e_i$ and consumption $c_i(R_T)$, $i = 1, \ldots, m$
  - State-dependent utility function

# State-Dependent Utility - Literature Review

- Axiomatisation
  - Dreze and Rustichini (2004)
  - Evans and Viscusi (1991)
  - Mas-Colell, Winston und Green (1995)
- Empirical evidence
  - Karni, Schmeidler and Vind (1983)

# Individual Preferences

- Consumer $i$'s extended expected utility, Mas-Colell et al. (1995)

$$U^i\left\{c_i\left(R_T\right)\right\} = \mathsf{E}\left[u^i\left\{R_T, c_i\left(R_T\right)\right\}\right],$$

with $u^i : \mathbb{R}^2_+ \to \mathbb{R}$ - state dependent utility index

$$u^i\left\{R_T, c_i\left(R_T\right)\right\} = u_i^0\left\{c_i\left(R_T\right)\right\}\mathbf{I}\left\{R_T \in [0, x_i]\right\} + u_i^1\left\{c_i\left(R_T\right)\right\}\mathbf{I}\left\{R_T \in (x_i, \infty)\right\}$$

$x_i \in [0, \infty)$ - reference point of consumer $i$; $x_1 \leq \cdots \leq x_m$
$u_i^0$, $u_i^1 : \mathbb{R}_+ \to \mathbb{R}$ - utility indices
- strictly increasing, concave and twice cts differentiable

# Individual Preferences

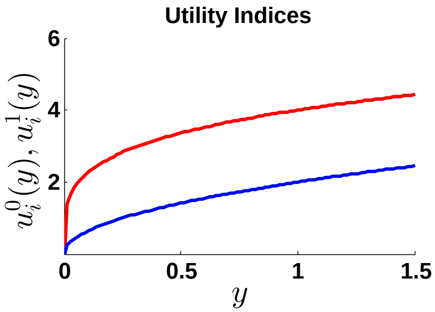


Figure 5: Utility indices $u_i^0(y) = y^{0.25}/0.25$ (bearish market) and $u_i^0(y) = y^{0.50}/0.50$ (bullish market)

# Equilibrium

- Individual optimization

$$\bar{c}_i\left(R_T\right) = \arg\max_{c_i\left(R_T\right)} U^i\left\{c_i\left(R_T\right)\right\}$$

$$\text{s.t. } \mathrm{E}\left[c_i(R_T)\mathcal{K}(R_T)\right] \le e_i$$

- Market clearing

$$\sum_{i=1}^{m} \bar{c}_i\left(R_T\right) = \sum_{i=1}^{m} e_i\left(R_T\right) \stackrel{\text{def}}{=} \bar{e}\left(R_T\right)$$

  - Pareto optimal $\bar{c}_1\left(R_T\right), \ldots, \bar{c}_m\left(R_T\right)$

# Aggregated Preferences

- Aggregated extended expected preferences

$$U_\alpha\left\{\bar{e}\left(R_T\right)\right\} = \mathsf{E}\left[u_\alpha\left\{R_T, \bar{e}\left(R_T\right)\right\}\right],$$

with $u_\alpha : \mathbb{R}^2_+ \to \mathbb{R}$ - aggregated indirect utility

$$\begin{aligned} u_\alpha\left\{R_T, \bar{e}\left(R_T\right)\right\} = & u_{\alpha,1}\left\{\bar{e}\left(R_T\right)\right\} \mathbf{I}\left\{R_T \in [0, x_1]\right\} + \\ & + \sum_{i=1}^{m-1} u_{\alpha,i+1}\left\{\bar{e}\left(R_T\right)\right\} \mathbf{I}\left\{R_T \in (x_i, x_{i+1}]\right\} + \\ & + u_{\alpha,m+1}\left\{\bar{e}\left(R_T\right)\right\} \mathbf{I}\left\{R_T \in (x_m, \infty)\right\} \end{aligned}$$

$$u_{\alpha,j}\left\{\bar{e}\left(R_T\right)\right\} = \sum_{k=1}^{m} \alpha_k u_k^0\left\{\bar{c}_k\left(R_T\right)\right\} \mathbf{I}\left\{k \geq j\right\} + \sum_{k=1}^{m} \alpha_k u_k^1\left\{\bar{c}_k\left(R_T\right)\right\} \mathbf{I}\left\{k < j\right\}$$

for $j = 1, \ldots, m+1$ and importance weights $\alpha = (\alpha_1, \ldots, \alpha_m)^\top$

# Pricing Kernel

**Theorem**

For every $\alpha_i > 0$ there exists $\beta_i$ s.t.

$$\alpha_i \beta_i \mathcal{K}(r_T) = \widetilde{\mathcal{K}}_\pi(r_T) = \left.\frac{\partial u_{\alpha,1}\{y\}}{\partial y}\right|_{y=r_T} \mathbf{I}\{r_T \in [0, x_1]\} + \\ + \sum_{i=1}^{m-1} \left.\frac{\partial u_{\alpha,i+1}\{y\}}{\partial y}\right|_{y=r_T} \mathbf{I}\{r_T \in (x_i, x_{i+1}]\} + \\ + \left.\frac{\partial u_{\alpha,m+1}\{y\}}{\partial y}\right|_{y=r_T} \mathbf{I}\{r_T \in (x_m, \infty)\}.$$

for every realization $r_T$ of $R_T$ and $\bar{e}(r_T) = r_T$ .

Note: $\widetilde{\mathcal{K}}_\pi(r_T)$ is nonincreasing separately on the intervals $[0, x_1], (x_1, x_2], \ldots, (x_m, \infty)$ but may be nonmonotone at $x_i$'s

# Example 1

**Example 1.** Consider $m$ investors with identical reference point $x_1$ that switch between constant relative risk aversion (CRRA) utilities $u_i^0(y) = y^{\gamma_i^0}/\gamma_i^0$ and $u_i^1(y) = y^{\gamma_i^1}/\gamma_i^1$, $0 < \gamma_i^0 < \gamma_i^1 < 1$.

$$\widetilde{\mathcal{K}}_\pi\left(r_T\right) = r_T^{\gamma_\alpha^0 - 1}\,\mathbf{I}\left\{r_T \in [0, x_1]\right\} + r_T^{\gamma_\alpha^1 - 1}\,\mathbf{I}\left\{r_T \in (x_1, \infty)\right\},$$

1-$\gamma_\alpha^\ell = r_T / \sum_{i=1}^{m} \frac{\bar{c}_i\left(r_T\right)}{\gamma_i^\ell - 1}$, $\ell = \{0, 1\}$ - implied CRRA coeff's

# Example 1

▸ R code

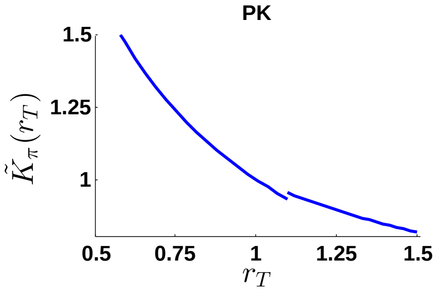


Figure 6: Pricing kernel $\widetilde{\mathcal{K}}_\pi\left(r_T\right)$ for $x_1 = 1.1$ and $\gamma_\alpha^0 = 0.25 < \gamma_\alpha^1 = 0.50$

# Example 2

**Example 2.** Consider $m$ investors with possibly different reference points $x_i$'s that switch between CRRA utilities $u^0(y) = b_0 \frac{y^\gamma}{\gamma}$ and $u^1(y) = b_1 \frac{y^\gamma}{\gamma}$. Let $F(r_T)$ be the cdf of the reference points

$$F(r_T) = m^{-1} \sum_{i=1}^{m} \mathbf{I}\{x_i \leq r_T\}$$

$$\mathcal{K}_{v,F}(r_T) = \widetilde{\mathcal{K}}_\pi(r_T) = \left[ \frac{r_T}{\{1 - F(r_T)\} b_0^{\frac{1}{1-\gamma}} + F(r_T) b_1^{\frac{1}{1-\gamma}}} \right]^{\gamma - 1} \qquad (1)$$

for parameters $v = (\gamma, b_0, b_1)^\top$, $0 < b_0 \leq b_1$

# Example 2 ▸ R code

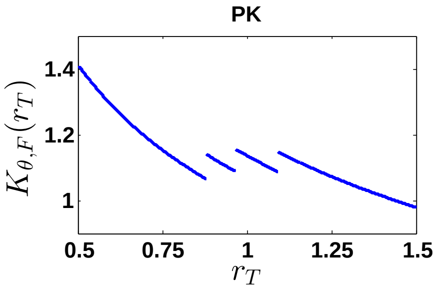


Figure 7: Pricing kernel $\mathcal{K}_{v,F}(r_T)$ with $\gamma = 0.5$, $b_0 = 1$, $b_1 = 1.2$ and $m = 3$ with uniformly generated reference points

# Example 2 ▸ R code

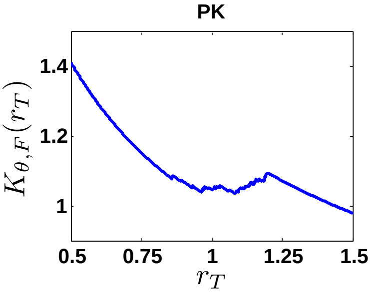


Figure 8: Pricing kernel $\mathcal{K}_{v,F}(r_T)$ with $\gamma = 0.5$, $b_0 = 1$, $b_1 = 1.2$ and $m = 40$ (left) and $m = 400$ (right) with reference points generated from a triangular distribution

# Example 2

R code

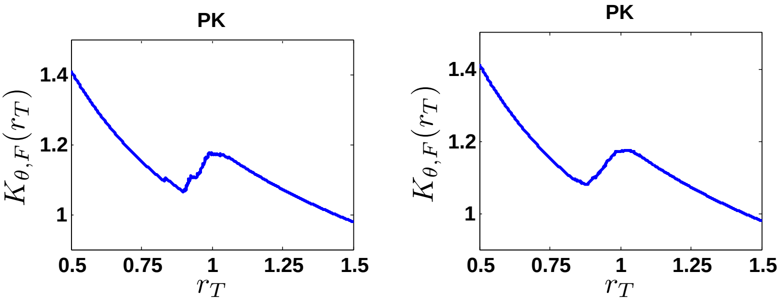


Figure 9: Pricing kernel $\mathcal{K}_{v,F}(r_T)$ with $\gamma = 0.5$, $b_0 = 1$, $b_1 = 1.2$ and $m = 40$ (left) and $m = 400$ (right) with reference points generated from a normal distribution $\mathrm{N}(0.95, 0.05)$

# Example 2

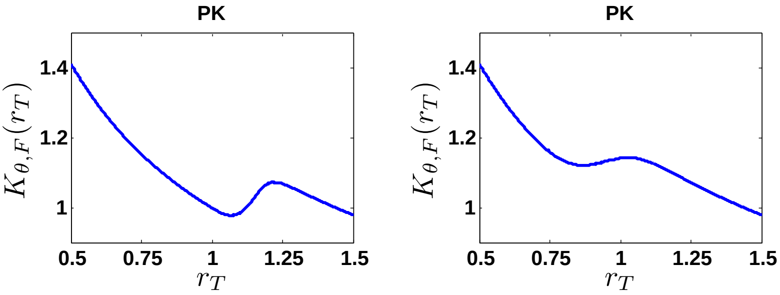


Figure 10: Pricing kernel $\mathcal{K}_{v,F}(r_T)$ with $\gamma = 0.5$, $b_0 = 1$, $b_1 = 1.2$ and $m = 4000$ with reference points generated from a normal distribution N$(1.15, 0.05)$ (left) and N$(0.95, 0.10)$ (right)

# Fitting EPK's

- Find $v$ and $F$ that minimize

$$\sum_{j=1}^{n} \left\{ \widehat{\mathcal{K}}(s_j) - \mathcal{K}_{v,F}(s_j) \right\}^2 \qquad (2)$$

for the estimate $\widehat{\mathcal{K}}$ at points $\{s_j\}_{j=0}^{n}$

$$\mathcal{K}_{v,F}(x) = \left[ \frac{x}{\{1 - F(x)\} b_0^{\frac{1}{\delta}} + F(x) b_1^{\frac{1}{\delta}}} \right]^{-\delta}$$

with $v = (\delta, b_0, b_1)^\top$, $\delta = 1 - \gamma$ and $F$ cdf.

# Parameters Identifiability

For $\delta, b_0, b_1 > 0$ and $b_0 \leq b_1$

$$x\mathcal{K}^{\frac{1}{\delta}}_{v,F}(x) = \{1 - F(x)\}\, b_0^{\frac{1}{\delta}} + F(x)\, b_1^{\frac{1}{\delta}} \qquad (3)$$

is a monotonically increasing function bounded between $b_0^{\frac{1}{\delta}}$ and $b_1^{\frac{1}{\delta}}$.

- For discrete reference points $v$ is identifiable ▸ Discrete RP
- For $F$ continuous $v$ is not identifiable

# Data

- Financial markets
  - EUREX European option data on 20000920 and 20060621
  - Daily DAX returns - past 500 observations until 20000920 and 20060621 respectively
- Pricing kernels
  - $\widehat{\mathcal{K}}(r_T)$ - Grith et al. (2010)
  - $\mathcal{K}_{v,F}(r_T) = \widetilde{\mathcal{K}}_{\pi}(r_T)$ - semi-parametric PK (1)
  - $\mathcal{K}_{\widehat{v},\widehat{F}}(r_T)$ - estimated $\mathcal{K}_{v,F}(r_T)$

# Fitting Results: Discrete RP

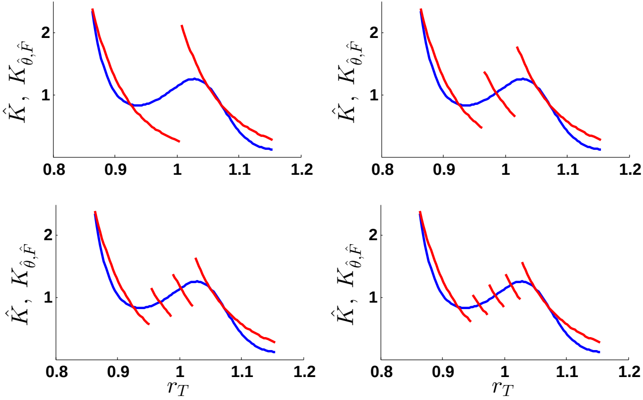


Figure 11: $\widehat{K}\,(r_T)$ on 20060621 and $\mathcal{K}_{\widehat{v},\widehat{F}}\,(r_T)$ for $m = 1, 2, 3, 4$.
$\widehat{v} = (-13.96, 0.27, 2.38)^\top$

# Continuous F: Parametric Case

Assume

$$F(x) = F_\phi(x) = \frac{T(x)^\phi}{\left[T(x)^\phi + \{1 - T(x)\}^\phi\right]^\phi}$$

$\phi > 0$ distortion parameters and $T$ sigmoid distribution

$$T(x) = [1 + \exp\{-a(x-c)\}]^{-1}$$

$a > 0$ and $c \in \mathbb{R}$. Then find $v$ and $F_\phi$ that minimize

$$\sum_{j=1}^{n} \left\{ \widehat{\mathcal{K}}(s_j) - \mathcal{K}_{v,F_\phi}(s_j) \right\}^2$$

# Fitting Results: Continuous $\widehat{F}_\phi$

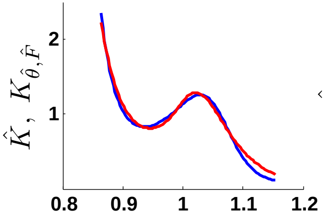


Figure 12: $\widehat{K}\,(r_T)$ on 20060621 and $\mathcal{K}_{\widehat{v},\widehat{F}_\phi}\,(r_T)$ (left) and $\widehat{F}_\phi\,(r_T)$ (right) for $\hat{\delta} = 21.10$ $\hat{b}_0 = 0.09$, $\hat{b}_1 = 3.99$, $\hat{a} = 65.01$, $\hat{c} = 0.97$, $\hat{\psi} = 0.58$

# Fitting Results: Continuous $\widehat{F}_\phi$

For fixed $\delta$ find $\hat{b}_0$, $\hat{b}_1$, $\hat{a}$, $\hat{c}$, $\hat{\psi}$ minimize 2

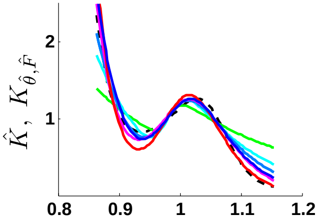


Figure 13: $\widehat{K}(r_T)$ and $\widehat{F}(r_T)$ on 20060621 for $\delta = 5$ (green), $\delta = 10$ (cyan), $\delta = 15$ (light blue), $\delta = 20$ (dark blue), $\delta = 25$ (magenta), $\delta = 30$ (red)

# Continuous F: Semi-Parametric Case

Assume

$$F(x) = \int_0^x \sum_{k=1}^P \beta_k \psi_k(u)du = \sum_{k=1}^P \beta_k \int_0^x \psi_k(u)du = \sum_{k=1}^P \beta_k \Psi_k(x)$$

For fixed $P$ and fixed $\delta$ find $(b_0, b_1, \beta_1, \dots, \beta_p)^\top$ that minimize

$$\sum_{j=1}^n \left\{ s_j \widehat{\mathcal{K}}(s_j) - \sum_{k=1}^P \beta_k \Psi_k(s_j)(b_1^{\frac{1}{\delta}} - b_0^{\frac{1}{\delta}}) + b_0^{\frac{1}{\delta}} \right\}^2$$

under the restriction that $F$ is a distribution.

# Conclusions

### Pricing kernel derivation

- Reference points determine jumps in the aggregate utility
- State-dependent preferences may explain the EPK paradox

### Fitting EPK's

- Quality increases with the number of switching points
- Fully parametric PK specification successfully applied

# Conclusions

**Further Research**

- Statistical estimation methodology for semi-parametric PK's
- Theoretical properties of $\widehat{v}$ and $\widehat{F}$
- Multidimensional reference points
- Dynamic implementation (PK's, reference points)

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# Risk Neutral Valuation

▸ Motivation

- Present value of the payoffs $\psi(S_T)$

$$P_0 = \mathsf{E}_Q\left[e^{-Tr}\psi(s_T)\right] = \int_0^\infty e^{-Tr}\psi(s_T)\,\mathcal{K}(s_T)p(s_T)\,ds_T$$

$r$ risk free interest rate, $\{S_t\}_{t\in[0,T]}$ stock price process,
$p$ pdf of $S_T$, $Q$ risk neutral measure, $\mathcal{K}(\cdot)$ pricing kernel

# PK under the Black-Scholes Model

Motivation

- Geometric Brownian motion for $S_t$

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t$$

$\mu$ mean, $\sigma$ volatility, $W_t$ Wiener process

- Physical density $p$ is log-normal, $\tau = T - t$

$$p_t(S_T) = \frac{1}{S_T\sqrt{2\pi\sigma^2\tau}} \exp\left[-\frac{1}{2}\left\{\frac{\log(S_T/S_t) - \left(\mu - \frac{\sigma^2}{2}\right)\tau}{\sigma\sqrt{\tau}}\right\}^2\right]$$

- Risk neutral density $q$ is log-normal: replace $\mu$ by $r$

# PK under the Black-Scholes Model

▸ Motivation

- PK is a decreasing function in $S_T$ for fixed $S_t$

$$\mathcal{K}(S_t, S_T) = \left(\frac{S_T}{S_t}\right)^{-\frac{\mu-r}{\sigma^2}} \exp\left\{\frac{(\mu-r)\left(\mu+r-\sigma^2\right)\tau}{2\sigma^2}\right\}$$
$$= b\left(\frac{S_T}{S_t}\right)^{-\delta}$$

$b = \exp\left\{\frac{(\mu-r)\left(\mu+r-\sigma^2\right)\tau}{2\sigma^2}\right\}$ and $\delta = \frac{\mu-r}{\sigma^2} \geq 0$ constant relative risk aversion (CRRA) coefficient

# Example 1

▸ Example 1

```
# Step 1/3: Input parameters
R = t(matrix(seq(0.8, 1.2, by = 0.01), 1))
x0 <- 1.1
gamma0 <- 0.25
gamma1 <- 0.50


# Step 2/3: Define the PK
K = R[R <= x0, ] ^ (gamma0 - 1)
K2 = R[R >= x0, ] ^ (gamma1 - 1)

# Step 3/3: Plot the PK against simple gross market return
plot(R[R <= x0, ], K, type = 'l', lwd = 3, col = "blue",
 xlim = c(0.8, 1.2), ylim = c(0.8, 1.25), xlab = "r_T")
lines(R[R >= x0, ], K2, type = 'l', lwd = 3, col = "blue",
 xlim = c(0.8, 1.2), ylim = c(0.8, 1.25), xlab = "r_T")
```

# Example 2

Example 2

```
gamma = 0.5
b0 = 1
b1 = 1.2

# Step 1/3: Input parameters and F_n
n = 1000
s = seq(0.5, 1.5, 0.2/n)
m = 10 # number of switching points
x = runif(m, 0.8, 1.2)
F_n = ecdf(x)(s)

# Step 2/3: Define the PK
PK = (s/((1 - F_n)*b0^(1/(1-gamma)) + F_n*b1^(1/(1-gamma))))^(gamma-1)

# Step 3/3: Plot the PK against simple gross market return
plot( cbind(s, PK) )
```

# Example 2

Example 2

```
gamma = 0.5
b0 = 1
b1 = 1.2

# Step 1/3: Input parameters and F_n
n = 1000
s = seq(0.5, 1.5, 0.2/n)
m = 40 # number of switching points
x = 0.8 + 0.4*sqrt(runif(m))
F_n = ecdf(x)(s)

# Step 2/3: Define the PK
PK = (s/((1 - F_n)*b0^(1/(1-gamma)) + F_n*b1^(1/(1-gamma))))^(gamma-1)

# Step 3/3: Plot the PK against simple gross market return
plot( cbind(s, PK) )
```

# Example 2

▸ Example 2

```
gamma = 0.5
b0 = 1
b1 = 1.2

# Step 1/3: Input parameters and F_n
n = 1000
s = seq(0.5, 1.5, 0.2/n)
m = 40 # number of switching points
F_n = pnorm( 20*(s-0.95) )

# Step 2/3: Define the PK
PK = (s/((1 - F_n)*b0^(1/(1-gamma)) + F_n*b1^(1/(1-gamma))))^(gamma-1)

# Step 3/3: Plot the PK against simple gross market return
plot( cbind(s, PK) )
```

# Discrete RP

▸ Parameters Identifiability

$$F\left(r_T\right) = m^{-1} \sum_{i=1}^{m} \mathbf{I}\left\{x_i \leq r_T\right\}$$

For $L$ distinct reference points $x_1 < x_2 < \ldots < x_L$, on any arbitrary interval $(x_{l-1}, x_l]$ with $l = 1, \ldots, L+1$

$$F(x) = F_L(x) = const. = c_l$$

Using (3)

$$x\mathcal{K}_{v,F}^{\frac{1}{\delta}}\left(x\right) = (1 - c_l) b_0^{\frac{1}{\delta}} + c_l b_1^{\frac{1}{\delta}} = const.,$$

which identifies $v$.